Indag. Mathem., N.S., 15 (4), 453–468 December 20, 2004

Smooth automorphisms and path-connectedness in Borel dynamics

by S. Bezuglyi [1], K. Medynets

Institute for Low Temperature Physics, 47 Lenin ave., 61103 Kharkov, Ukraine

Communicated by Prof. M.S. Keane at the meeting of September 27, 2004

ABSTRACT

Let $Aut(X,\mathcal{B})$ be the group of all Borel automorphisms of a standard Borel space $(X,\mathcal{B})$. We study topological properties of $Aut(X,\mathcal{B})$ with respect to the uniform and weak topologies, τ and p, defined in [Bezuglyi S., Dooley A.H., Kwiatkowski J., Topologies on the group of Borel automorphisms of a standard Borel space, Preprint, 2003]. It is proved that the class of smooth automorphisms is dense in $(Aut(X,\mathcal{B}),p)$. Let $Ctbl(X)$ denote the group of Borel automorphisms with countable support. It is shown that the topological group $Aut_0(X,\mathcal{B}) = Aut(X,\mathcal{B})/Ctbl(X)$ is path-connected with respect to the quotient topology τ_0. It is also proved that $Aut_0(X,\mathcal{B})$ has the Rokhlin property in the quotient topology p_0, i.e., the action of $Aut_0(X,\mathcal{B})$ on itself by conjugation is topologically transitive.

0. INTRODUCTION

In the present paper, we continue the study of topological properties of the group $Aut(X,\mathcal{B})$ of all Borel automorphisms of a standard Borel space $(X,\mathcal{B})$. We consider two topologies, τ and p, on $Aut(X,\mathcal{B})$ which take their origins in ergodic theory. They were defined and studied in the context of Borel and Cantor dynamics in [3–5,7,8]. Recall that the topology τ is defined by the base of neighborhoods $U(T;\mu_1,\dots,\mu_n;\varepsilon) = \{S \in Aut(X,\mathcal{B}) \mid \mu_i(E(S,T)) < \varepsilon,\ i = 1,\dots,n\}$, where $\mu_1,\dots,\mu_n$ are Borel probability measures on X and $E(S,T) = \{x \in X\colon\ Tx \neq Sx\} \cup \{x \in X\colon\ S^{-1}x \neq T^{-1}x\}$. Obviously, τ is a direct analogue of the well-known uniform topology on the group $Aut(X,\mathcal{B},\mu)$ of all non-singular automorphisms of a measure space generated by the metric $d(S,T) = \mu(E(S,T))$.

E-mails: bezuglyi@ilt.kharkov.ua (S. Bezuglyi), medynets@ilt.kharkov.ua (K. Medynets).

[1] Supported in part by CRDF grant UM1-2546-KH-03.

It is worthwhile to mention that, in fact, $Aut(X, \mathcal{B}, \mu)$ is formed by classes of automorphisms coinciding μ-almost everywhere. It allows one to neglect the behavior of automorphisms on sets of zero measure. Topological properties of $(Aut(X, \mathcal{B}, \mu), d)$ were extensively studied in ergodic theory (see, for example, [1, 6,9,10,13,18]). The other topology, p, on $Aut(X, \mathcal{B})$ is defined by neighborhoods $W(T; F_1, \ldots, F_n) = \{S \in Aut(X, \mathcal{B}) \mid SF_i = TF_i,\ i = 1, \ldots, n\}$ where $F_1, \ldots, F_n$ are Borel sets. It was shown in [3] that p can be treated as an analogue of the weak topology d_w which has been also widely used in ergodic theory. Note also that, in the context of Cantor dynamics, p is equivalent to the sup-topology on the set of all homeomorphisms. Based on this observation, we call τ and p the uniform and weak topologies on $Aut(X, \mathcal{B})$ respectively.

Our goal is to find out which of topological properties known in ergodic theory for $Aut(X, \mathcal{B}, \mu)$ hold for $Aut(X, \mathcal{B})$ with respect to the topologies τ and p. For instance, it is important for many applications to know dense subsets in $Aut(X, \mathcal{B})$ which consist of "relatively simple" Borel automorphisms. It was shown in [3] that the set $\mathcal{P}er$ of periodic automorphisms is dense in $(Aut(X, \mathcal{B}), \tau)$ but non-dense in $(Aut(X, \mathcal{B}), p)$. Therefore one needs to extend $\mathcal{P}er$ to produce a dense subset in $Aut(X, \mathcal{B})$ with respect to p. The class $\mathcal{S}$ of smooth Borel automorphisms is a natural extension of periodic automorphisms. By definition, T is smooth if there exists a Borel subset in X which intersects every T-orbit exactly once. In this paper (see Section 2), we prove that the set of smooth Borel automorphisms is dense in $(Aut(X, \mathcal{B}), p)$. This statement has been conjectured by A. Kechris.

A number of papers in ergodic theory was devoted to the study of connectedness of $Aut(X, \mathcal{B}, \mu)$ in the weak and uniform topologies (see, for example, [6,10,13, 14]). In particular, it was proved that $Aut(X, \mathcal{B}, \mu)$ was path-connected and even contractible. It is not hard to show that $Aut(X, \mathcal{B})$ is not path-connected in τ because there is no continuous path connecting the identity with an automorphism with countable support. At first sight, the situation seems to be different in Borel dynamics. But if we factorize $Aut(X, \mathcal{B})$ (like in ergodic theory) by a closed normal subgroup consisting of automorphisms whose behavior can be neglected, we do produce a path-connected quotient group. From this point of view, it is natural to say that $S, T \in Aut(X, \mathcal{B})$ are equivalent if they are different on an at most countable set. If $Ctbl(X)$ denotes the group of all Borel automorphisms with countable support, then $Aut_0(X, \mathcal{B}) = Aut(X, \mathcal{B})/Ctbl(X)$ is a Hausdorff topological group with respect to the quotient topologies τ_0 and p_0. Note that the group $Ctbl(X)$ was also considered in [19]. It turns out that such a factorization improves topological properties $Aut(X, \mathcal{B})$ in both quotient topologies τ_0 and p_0. Namely, the group $(Aut_0(X, \mathcal{B}), \tau_0)$ becomes path-connected and $(Aut_0(X, \mathcal{B}), p_0)$ possesses the so called *Rokhlin property*, i.e., the action of $Aut_0(X, \mathcal{B})$ on itself by conjugation is topologically transitive (see [12,5] for the Rokhlin property in Cantor dynamics). In fact, we prove an even stronger result by showing that the conjugacy class of every aperiodic smooth automorphism is dense.

Most definitions and notions used in this paper are mostly taken from the book [17]. We collected in Section 1 definitions and facts which are used in the paper. When we say that T is an automorphism of $(X, \mathcal{B})$, we always mean that T

is a Borel automorphism. We will also use the term "automorphism" for elements of the quotient group $Aut_0(X, \mathcal{B})$.

1. PRELIMINARIES

1.1. Let $(X, \mathcal{B})$ be a *standard Borel space* with the σ-algebra of Borel sets $\mathcal{B}$. This means, by definition, that $(X, \mathcal{B})$ is (Borel) isomorphic to a Polish space, i.e., a complete separable metric space. Recall several facts about standard Borel spaces:

(i) any two standard Borel spaces are Borel isomorphic;
(ii) if $A \in \mathcal{B}$, then A is either at most countable or has the cardinality continuum;
(iii) if Borel sets A, B have the same cardinality, then they are isomorphic.

1.2. Denote by $Aut(X, \mathcal{B})$ the group of all Borel automorphisms of $(X, \mathcal{B})$. Let $T \in Aut(X, \mathcal{B})$ and $A \in \mathcal{B}$. The set $\bigcup_{n\in\mathbb{Z}} T^n A$ is called the *saturation* of A with respect to T and denoted by $s_T A$ (or simply sA if T is clear from the context). A Borel set W is said to be *wandering* with respect to T if $T^n W \cap W = \emptyset$, $n \in \mathbb{N}$. A Borel set $A \subset X$ is called a *complete section* with respect to T (or simply a T-section) if every T-orbit meets A at least once, i.e., $s_T A = X$. A point x from a Borel set A is called *recurrent* with respect to T if there exists $n \in \mathbb{N}$ such that $T^n x \in A$.

1.3. Denote by $\mathcal{A}p$ and $\mathcal{P}er$ the sets of aperiodic and periodic automorphisms respectively.

We say that a transformation $T \in Aut(X, \mathcal{B})$ is *smooth* if there exists a complete Borel section A such that A meets every T-orbit exactly once. We will denote the class of smooth automorphisms by $\mathcal{S}$. Obviously, any periodic Borel automorphism is smooth. On the other hand, if X is a compact metric space and T is an aperiodic homeomorphism of X, then T cannot be smooth.

1.4. We will use the following basic statements taken from [17].

(a) (*Poincaré recurrence lemma.*) Let $T \in Aut(X, \mathcal{B})$ and $A \in \mathcal{B}$. Then there exists a wandering set $W \subset A$ such that for each $x \in A - \bigcup_{k\in\mathbb{Z}} T^k W$ the point $T^n x$ belongs to A for infinitely many positive n and also for infinitely many negative n.
(b) Let $T \in Aut(X, \mathcal{B})$. Then X can be partitioned into a disjoint union of Borel sets $X = X_\infty \cup \bigcup_{k\geqslant 1} X_k$ where points from X_k, $k < \infty$, have period k and X_∞ consists of aperiodic points.
(c) Let $T \in Aut(X, \mathcal{B})$ and let X_k, $k < \infty$, be as in (b). Then there exists a Borel set $B_k \subset X_k$ such that $X_k = \bigcup_{i=0}^{k-1} T^i B_k$ and the union is disjoint.

1.5. T-towers. Let $T \in Aut(X, \mathcal{B})$. Assume that all points from A are recurrent with respect to T. For $x \in A$, define $n(x) = n_A(x)$ as the smallest positive integer such that $T^{n(x)}x \in A$ and $T^i x \notin A$, $0 < i < n(x)$. Let $C_k = \{x \in A \mid n_A(x) = k\}$,

$k \in \mathbb{N}$ (some of the C_k's may be empty). Notice that $T^k C_k \subset A$ and $\xi_k = \{T^i C_k \mid i = 0, \ldots, k-1\}$ consists of pairwise disjoint sets. We call ξ_k a *T-tower* with base C_k and top $T^{k-1}C_k$. The set $T^i C_k$ is called the ith level of ξ_k. The height of ξ_k is k.

Since $T^n x \in A$ for infinitely many positive and negative n, we have

$$\bigcup_{n \geqslant 0} T^n A = \bigcup_{n \in \mathbb{Z}} T^n A = s_T A$$

and

$$\bigcup_{n \geqslant 0} T^n A = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} T^i C_k.$$

The above relation shows that $\xi = \{\xi_k \colon k \in \mathbb{N}\}$ forms a partition of $s_T A$ into T-towers ξ_k, $k \in \mathbb{N}$. Notice that T maps the union of tops of these towers onto the union of their bases.

Given a partition ξ on X, a Borel set $B \subset X$ is called a ξ-set if it is a union of atoms of ξ.

1.6. The next lemma is one of the main tools in the study of Borel automorphisms. It is used in various problems related to finding a suitable approximation of an aperiodic automorphism, in particular, in the proof of the Rokhlin lemma [15,17, 3].

1.7. Lemma. *Let $T \in Aut(X, \mathcal{B})$ be an aperiodic Borel automorphism of a standard Borel space $(X, \mathcal{B})$, i.e., $X = X_\infty$. Then there exists a sequence (A_n) of Borel sets such that*

(i) $X = A_0 \supset A_1 \supset A_2 \supset \cdots$,
(ii) $\bigcap_n A_n = \emptyset$,
(iii) *A_n and $X \setminus A_n$ are complete T-sections, $n \in \mathbb{N}$,*
(iv) *for $n \in \mathbb{N}$, every point in A_n is recurrent,*
(v) *for $n \in \mathbb{N}$, $A_n \cap T^i(A_n) = \emptyset$, $i = 1, \ldots, n-1$,*
(vi) *the base $C_k(n)$ of every non-empty T-tower constructed over A_n is an uncountable Borel set, $n \in \mathbb{N}$.*

For the proof, see [2, Lemma 4.5.3] where (i)–(iii) have been proved in more general settings of countable Borel equivalence relations. It is shown in [17, Chapter 7] that one can refine the choice of (A_n) to get (iv) and (v). It is clear that one can remove an at most countable set of points from each A_n to satisfy property (vi).

A sequence of Borel sets (A_n) satisfying **1.7** is called a *vanishing sequence of markers*.

1.8. Recall now the definition of the uniform and weak topologies on $Aut(X, \mathcal{B})$ following [3]. Let $\mathcal{M}_1(X)$ denote the set of all Borel probability measures on X.

A measure $\mu \in \mathcal{M}_1(X)$ is called continuous (non-atomic) if $\mu(\{x\}) = 0$ for all $x \in X$. The Dirac measure at $x \in X$ is denoted by δ_x. For $T, S \in Aut(X, \mathcal{B})$, define $E(S,T) = \{x \in X\colon\ Tx \neq Sx\} \cup \{x \in X\colon\ S^{-1}x \neq T^{-1}x\}$.

1.9. Definition. The topologies τ and p on $Aut(X, \mathcal{B})$ are defined, respectively, by the bases of neighborhoods $\mathcal{U} = \{U(T; \mu_1, \ldots, \mu_n; \varepsilon)\}$ and $\mathcal{W} = \{W(T; F_1, \ldots, F_n)\}$ where

$$U(T; \mu_1, \mu_2, \ldots, \mu_n; \varepsilon) = \big\{S \in Aut(X, \mathcal{B}) \mid \mu_i\big(E(S,T)\big) < \varepsilon,\ i = 1, \ldots, n\big\},$$
$$W(T; F_1, F_2, \ldots, F_n) = \big\{S \in Aut(X, \mathcal{B}) \mid SF_i = TF_i,\ i = 1, \ldots, n\big\}.$$

Here $T \in Aut(X, \mathcal{B})$, $\mu_1, \ldots, \mu_n \in \mathcal{M}_1(X)$, $\varepsilon > 0$, and $F_1, \ldots, F_n \in \mathcal{B}$.

It was shown in [3] that $Aut(X, \mathcal{B})$ is a Hausdorff topological group with respect to these topologies. More topological properties of $Aut(X, \mathcal{B})$ and its subsets can be found in [3].

1.10. Remark. If in the definition of τ one takes the set $E_0(T, S) = \{x \in X\colon\ Sx \neq Tx\}$, then the obtained new topology is, in fact, equivalent to τ. The proof of this fact is straightforward.

1.11. Let $Ctbl(X)$ be defined as a subset of $Aut(X, \mathcal{B})$ consisting of all automorphisms with countable support, that is $T \in Ctbl(X)$ if $|\{x \in X\colon\ Tx \neq x\}| \leqslant \aleph_0$ where $|A|$ denotes the cardinality of A. Note that $Ctbl(X)$ is a normal subgroup closed with respect to the topologies τ and p, see the proposition below. Therefore $Aut_0(X, \mathcal{B}) = Aut(X, \mathcal{B})/Ctbl(X)$ is a Hausdorff topological group with respect to the quotient topologies τ_0 and p_0. Considering elements from $Aut_0(X, \mathcal{B})$, we will identify Borel automorphisms which are different on a countable set, that is $S \sim S'$ if $|E(S, S')| \leqslant \aleph_0$. In other words, $S \sim S'$ if there exists $P \in Ctbl(X)$ such that $S = S'P$. This identification corresponds to the well-known approach used in measurable dynamics when two automorphisms are also identified if they are different on a set of measure 0.

1.12. Proposition. *$Ctbl(X)$ is a normal closed subgroup in $Aut(X, \mathcal{B})$ with respect to the topologies τ and p.*

Proof. It is obvious that $Ctbl(X)$ is a normal subgroup in $Aut(X, \mathcal{B})$, so it is enough to prove that it is closed in τ and p. To do this, suppose that there exists an automorphism $S \in \overline{Ctbl(X)}^{\tau} \setminus Ctbl(X)$. Then for any neighborhood $U(S) = U(S; \mu_1, \ldots, \mu_n; \varepsilon)$ there exists an automorphism $R \in U(S) \cap Ctbl(X)$, that is $\mu_i(E(R, S)) < \varepsilon$ for all i. Since $S \notin Ctbl(X)$, we have that $E = \{x \in X\colon\ Sx \neq x\}$ is uncountable. Let ν be a continuous measure on X such that $\nu(X \setminus E) = 0$. Consider a neighborhood $U_1 = U(S; \nu; \varepsilon)$ of S. Then for $R \in U_1 \cap Ctbl(X)$ we have that

$$\nu\big(\{x \in X\colon\ Sx \neq x\}\big) = 1, \qquad \nu\big(\{x \in X\colon\ Sx \neq Rx\}\big) < \varepsilon.$$

But $\nu(\{x \in X\colon Rx \neq x\}) = 0$, therefore $\nu(\{x \in X\colon Sx = x\}) = 0$ and $\nu(\{x \in X\colon Sx = Rx = x\}) > 1 - \varepsilon$, a contradiction.

Assume now that $S \in \overline{Ctbl(X)}^p \setminus Ctbl(X)$. Then $E = \{x \in X\colon Sx \neq x\}$ is uncountable and S-invariant. Let E_1 be an uncountable Borel subset of E such that $SE_1 \cap E_1 = \emptyset$. Then for every $R \in W(S; E_1)$ we have that $RE_1 \cap E_1 = \emptyset$. In particular, if $R \in W(S; E_1) \cap Ctbl(X)$, then we obtain that R acts non-trivially on the uncountable set E_1. This contradicts the definition of $Ctbl(X)$. $\square$

The bases of neighborhoods for τ_0 and p_0 consists of the sets $U_0(T; \mu_1, \ldots, \mu_n; \varepsilon) = U(T; \mu_1, \ldots, \mu_n; \varepsilon)Ctbl(X)$ and $W_0(T; F_1, \ldots, F_m) = W(T; F_1, \ldots, F_m) \times Ctbl(X)$, respectively. The next proposition shows that, in fact, τ_0 and p_0 are generated by neighborhoods U_0 and W_0 with continuous measures μ_i and uncountable sets F_j.

1.13. Proposition. *Given a τ_0-neighborhood $U_0 = U_0(T; \mu_1, \ldots, \mu_n; \varepsilon)$ and a p_0-neighborhood $W_0 = W_0(T; F_1, \ldots, F_m)$, there exist neighborhoods $U_0'(T; \nu_1, \ldots, \nu_n; \varepsilon) = U'(T; \nu_1, \ldots, \nu_n; \varepsilon)Ctbl(X)$ and $W_0'(T; B_1, \ldots, B_m) = W'(T; B_1, \ldots, B_m) \times Ctbl(X)$ of τ_0 and p_0, respectively, such that $U_0' \subset U_0$, $W_0' \subset W_0$ where measures $\nu_1, \ldots, \nu_n$ are continuous and Borel sets $B_1, \ldots, B_m$ are uncountable.*

Proof. Consider the countable set $A = \bigcup_{i=1}^n A_i$ where $A_i = \{x \in X\colon \mu_i(\{x\}) > 0\}$. Let $c_i = \mu_i(A)$ and assume that $c_i < 1$, $i = 1, \ldots, n$. Define

$$\nu_i(B) = \frac{\mu_i(B \cap A^c)}{\mu_i(A^c)}, \quad B \in \mathcal{B},\ i = 1, \ldots, n,$$

where $A^c := X \setminus A$. Clearly, ν_i is a non-atomic Borel probability measure on X. It remains to show that $U_0' = U'(T; \nu_1, \ldots, \nu_n; \varepsilon)Ctbl(X)$ is a subset of U_0. To do this, it suffices to check that for every $S \in U'(T; \nu_1, \ldots, \nu_n; \varepsilon)$ there exists $S_1 \in U(T; \mu_1, \ldots, \mu_n; \varepsilon)$ such that $S \sim S_1$. Let Γ be the countable group of automorphisms of X generated by T and S. Let $D = s_\Gamma A$ be the Γ-orbit of A. Define

$$S_1 x = \begin{cases} Tx, & x \in D, \\ Sx, & x \in D^c. \end{cases}$$

Obviously, $S_1 \sim S$. Since $E(T, S_1) \subset D^c \subset A^c$ and $E(T, S_1) \subset E(T, S)$, we have that for $i = 1, \ldots, n$,

$$\mu_i\big(E(T, S_1)\big) = \mu_i(A^c)\nu_i\big(E(T, S_1)\big) \leqslant \mu_i(A^c)\nu_i\big(E(T, S)\big) < \varepsilon\mu_i(A^c) < \varepsilon.$$

Notice that if $\mu_i(A) = 1$ for some i, then for any $S \in Aut(X, \mathcal{B})$ there exists $S_1 \sim S$ such that $\mu_i(E(S_1, T)) = 0$.

The proof for the topology p_0 is similar. $\square$

2.1. In this section, we prove that the p-closure of the set $\mathcal{S}$ of smooth automorphisms is the entire group $Aut(X, \mathcal{B})$. Moreover, it is shown that $(Aut_0(X, \mathcal{B}), p_0)$ has the Rokhlin property.

2.2. Theorem. $\bar{\mathcal{S}}^p = Aut(X, \mathcal{B})$. *Moreover, each p-neighborhood of an aperiodic automorphism necessarily contains an aperiodic smooth automorphism.*

Proof. Let $T \in Aut(X, \mathcal{B})$. Obviously, it suffices to consider the case when T is aperiodic. Take a p-neighborhood $W = W(T; F_1, F_2, \dots, F_n)$ of T. Without loss of generality, we can assume that the sets $\{F_1, F_2, \dots, F_n\}$ form a partition of X. Show that there exists a smooth aperiodic automorphism $S \in W$.

For $i = 1, 2, \dots, n$, consider the Borel sets

$$F_i^j = F_i \cap T F_j, \quad j = 1, 2, \dots, n.$$

Suppose that the collection $\{F_i^j\}_{i,j=1}^n$ contains exactly $q \leqslant n^2$ non-empty sets. Denote them by V_l, $l = 1, \dots, q$. Then $\{V_1, \dots, V_q\}$ is a partition of X which refines $\{F_1, \dots, F_n\}$.

Let

$$\begin{aligned} B_1 &= V_1, \\ B_2 &= V_2 - sB_1, \\ &\dots \\ B_q &= V_q - s(B_1 \cup B_2 \cup \dots \cup B_{q-1}). \end{aligned}$$

Here and below s stands for s_T. Clearly, $\{sB_1, sB_2, \dots, sB_q\}$ is a partition of X into Borel sets. Without loss of generality, we may assume that all sets $B_1, \dots, B_q$ are non-empty. For each B_i, find a wandering subset $A_i \subset B_i$ (see **1.4**) such that all points from $D_i = B_i - sA_i$ are recurrent. Therefore, by **1.5**, we can find a partition Ξ_i of sD_i into pairwise disjoint T-towers $\{\xi_i(k) \colon k \in \mathbb{N}\}$ such that the union of bases of these towers is D_i. For short, we will write

$$sD_i = \bigcup_{\xi \in \Xi_i} \xi, \quad i = 1, \dots, q.$$

For a tower ξ, denote by B_ξ and h_ξ its base and height, respectively. By construction, we have that

$$\bigcup_{\xi \in \Xi_i} B_\xi = D_i \subset V_i \subset F_m$$

and

$$\bigcup_{\xi \in \Xi_i} T^{h_\xi - 1} B_\xi = T^{-1} D_i \subset T^{-1} V_i = T^{-1}(F_k \cap T F_l) \subset F_l$$

for some $1 \leqslant m, l \leqslant n$.

Since each sB_i, $i = 1, \dots, q$, can be represented as a disjoint union of sA_i and sD_i, we obtain the partition

$$X = \bigcup_{i=1}^{q} sB_i = \bigcup_{i=1}^{q} sA_i \cup \bigcup_{i=1}^{q} sD_i.$$

The automorphism T restricted to $s\bigcup_{s=1}^{q} A_i = \bigcup_{i=1}^{q} sA_i$ is smooth since $\bigcup_{i=1}^{q} A_i$ is a wandering set for T. Define $S = T$ on $s\bigcup_{s=1}^{q} A_i$. To complete the proof, we need to define S on $\bigcup_{i=1}^{q} sD_i$ such that $S(sD_i \cap F_j) = T(sD_i \cap F_j)$ for all $j = 1, \dots, n$, $i = 1, \dots, q$.

Fix the set sD_i and let $\Xi_i^0 = \{\xi \in \Xi_i \colon |B_\xi| \leqslant \aleph_0\}$. Then

$$X_i^0 = \bigcup_{\xi \in \Xi_i^0} sB_\xi$$

is countable and therefore T, restricted to X_i^0, is smooth. Set $S = T$ on X_0^i.

For $\xi \in \Xi_i$, denote by $\xi' := \{T^j(B_\xi - X_0^i)\colon 0 \leqslant j \leqslant h_\xi - 1\}$. Setting $\Xi_i' = \{\xi' \colon \xi \in \Xi_i\}$, we get a disjoint union

$$sD_i = X_0^i \cup \bigcup_{\xi' \in \Xi_i'} \xi'.$$

Note that the cardinality of each tower $\xi' \in \Xi_i'$ is continuum.

Now, we define the automorphism S on each tower $\xi' \in \Xi_i'$ such that $S(\xi') = \xi'$ and S coincides with T on each level of the tower ξ' except the top of ξ'. To do this, we write down the base $B_{\xi'}$ of ξ' as a disjoint union $B_{\xi'} = \bigcup_{m \in \mathbb{Z}} B_{\xi'}(m)$ with uncountable Borel sets $B_{\xi'}(m)$. Let R_m be an arbitrary Borel isomorphism between $T^{h_{\xi'}-1}B_{\xi'}(m)$ and $B_{\xi'}(m+1)$, $m \in \mathbb{Z}$ (one can assume that $h_{\xi'} \geqslant 2$ by [17, Theorem 7.25]). Define $Sx = Tx$ for $x \in \{T^j B_{\xi'}(m)\colon 0 \leqslant j \leqslant h_\xi - 2, m \in \mathbb{Z}\}$ and $Sx = R_m x$ for $x \in T^{h_\xi - 1} B_\xi(m)$, $m \in \mathbb{Z}$. Then S is defined everywhere on X. To prove that $S \in W$, we need to show that $SF_j = TF_j$. Write down $F_j \cap sD_i$ as a disjoint union of sets E_0 and E_1 where $E_1 = \{x \in F_j \cap sD_i\colon x \in T^{h_\xi - 1}B_\xi$, for some $\xi \in \Xi_i\}$ and $E_0 = (F_j \cap sD_i) - E_1$. It follows from the construction that if $E_1 \neq \emptyset$, then $E_1 \subset F_j$ and $E_1 = \bigcup_{\xi \in \Xi_i} T^{h_\xi - 1} B_\xi = T^{-1} D_i$. Therefore, by definition of S, we have that $SE_1 = TE_1$. It is clear that $S = T$ on E_0. The proof is complete. □

2.3. Remark. (1) We note that the set $\mathcal{S}$ is dense in $(Aut(X, \mathcal{B}), \tau)$. It follows from the fact that the set of periodic automorphisms is dense in $(Aut(X, \mathcal{B}), \tau)$ [3, Corollary 2.6]. On the other hand, $\mathcal{S} \cap \mathcal{A}p$ is nowhere dense in $(Aut(X, \mathcal{B}), \tau)$ by [3, Theorem 2.8].

(2) The set of aperiodic smooth automorphisms is not dense in $(Aut(X, \mathcal{B}), p)$ because $\mathcal{A}p$ is a closed subset in $(Aut(X, \mathcal{B}), p)$ [3, Theorem 2.8].

2.4. Remark. B. Miller proved the following result [16] which may be used to simplify the proof of Theorem 2.2:

Suppose that X is a Polish space, $T : X \to X$ is a Borel automorphism, and $\{A_n\}_{n\in\mathbb{N}}$ is a partition of X into Borel sets. Then there is a countable T-invariant set $C \subset X$ and an aperiodic smooth Borel automorphism $S : X \setminus C \to X \setminus C$ such that $T(A_n \setminus C) = S(A_n \setminus C)$ for every $n \in \mathbb{N}$.

2.5. Proposition. *$\mathcal{P}er$ is a closed nowhere dense subset of $Aut(X, \mathcal{B})$ with respect to p.*

Proof. We first show that the set $\mathcal{P}er$ is closed. If we assume that there exists an automorphism $T \in \overline{\mathcal{P}er}^p \setminus \mathcal{P}er$, then T must have an aperiodic point $x_0 \in X$. Define $F = \{x_0\} \cup \{Tx_0\} \cup \{T^2x_0\} \cup \cdots$ and consider the p-neighborhood $W = W(T; F)$. Then W necessarily contains a periodic automorphism P such that $PF = TF$. Since $TF \subsetneqq F$, we obtain that $P^n F \subsetneqq F$ for all $n \in \mathbb{N}$. Therefore the point $\{x_0\} = F \setminus PF$ must be aperiodic for P, a contradiction.

Let $W(P) = W(P; F_1, \dots, F_n)$ be a p-neighborhood of a periodic automorphism P. We can assume that the sets $(F_1, \dots, F_n)$ form a partition of X. We will first show that $W(P)$ contains a non-periodic automorphism. By **1.4**, X is partitioned into P-towers $\xi_k = \{B_k, \dots, P^{k-1}B_k\}$ such that P has period k on the set $X_k = B_k \cup \cdots \cup P^{k-1}B_k$. One can refine the partition $\xi = (\xi_k \colon k \in \mathbb{N})$ to produce a new partition ξ' such that every $F_i, i = 1, \dots, n$, is a ξ'-set. Let $(B, \dots, T^{m-1}B)$ be a P-tower from ξ' with uncountable base. As in **2.2**, we can find an aperiodic automorphism T defined on $C = \bigsqcup_{i=0}^{m-1} T^i B$ such that $T(P^i B) = P^{i+1}B$, $i = 0, \dots, n-2$, and $T(P^{n-1}B) = B$. Define T on $X \setminus C$ by setting $T = P$. We see that $TF_i = PF_i$ for all i, i.e., $T \in W(P)$. It is clear that there exists a Borel set $F \subset C$ such that $TF \subsetneqq F$. Then $W_1 = W(T; F)$ contains no periodic automorphism. Thus, $(W(T) \cap W(P)) \cap \mathcal{P}er = \emptyset$ and we are done. □

2.6. In contrast to **2.3**, the situation for the quotient group $Aut_0(X, \mathcal{B})$ is different. It turns out that the set $\mathcal{S} \cap \mathcal{A}p$ is a dense subset in $(Aut_0(X, \mathcal{B}), p_0)$.

2.7. Theorem. *The set of aperiodic smooth automorphisms is dense in $(Aut_0(X, \mathcal{B}), p_0)$.*

Proof. By **2.2**, we only need to show that each neighborhood $W_0 = W_0(P; F_1, \dots, F_n)$ of $P \in \mathcal{P}er$ contains an aperiodic smooth automorphism S. It follows from **1.4** that X is decomposed into an at most countable collection of P-invariant towers $\Xi = \{\xi_k \colon k \in \mathbb{N}\}$. By **1.13**, we can assume that all ξ_k's are uncountable Borel sets. Let Ξ' be a refinement of Ξ, obtained by cutting the towers from Ξ, such that each F_i is a Ξ'-set.

We partition each $\xi \in \Xi'$ into a disjoint union $\xi = \bigsqcup_{m\in\mathbb{Z}} \xi_m$ of P-towers ξ_m such that the base of ξ_m is uncountable. To define S on ξ, we apply the method used in the proof of **2.2**. For fixed ξ_m, $m \in \mathbb{Z}$,, we set $S = P$ everywhere except the top of ξ_m and set $S = R_m$ on the top where R_m is a Borel isomorphism mapping the top of ξ_m onto the base of ξ_{m+1}. Then S is defined everywhere on X and is aperiodic. Note that every $\xi \in \Xi'$ is S-invariant. Since all towers ξ_m are of the same height, we have that S maps ξ-atoms onto themselves. It follows from this observation that $SF_i = PF_i$, $i = 1, \dots, n$, that is $S \in W_0$. □

2.8. Corollary. $\overline{\mathcal{A}p}^{p_0} = Aut_0(X, \mathcal{B})$.

Proof. This result is an easy consequence of **2.7**. □

2.9. The next statement proves a Borel version of the Rokhlin property for $(Aut_0(X, \mathcal{B}), p_0)$. Note that this property was considered in the settings of measurable and Cantor dynamics in [11,12,5,18].

2.10. Corollary (the Rokhlin property). *The action of $Aut_0(X, \mathcal{B})$ on itself by conjugation is transitive with respect to the topology p_0. Moreover, $Aut_0(X, \mathcal{B}) = \overline{\{T^{-1}ST\colon T \in Aut_0(X, \mathcal{B})\}}^{p_0}$ holds for any $S \in \mathcal{S} \cap \mathcal{A}p$.*

Proof. To prove this result it suffices to use **2.8** together with the simple fact that any two aperiodic smooth automorphisms are conjugate. □

2.11. The famous Rokhlin lemma on approximation aperiodic automorphisms in the uniform topology was proved in the context of Borel dynamics in [3,15,17]. We formulate here this statement in the following form.

2.12. Proposition. *Let T be an aperiodic Borel automorphism of a standard Borel space $(X, \mathcal{B})$ and let $\mu_1, \dots, \mu_k \in \mathcal{M}_1(X)$, $\varepsilon > 0$, and $n, m \geqslant 2$. Then there exists a Borel partition of X into T-towers $\Xi = \{\xi_k\colon k \in \mathbb{N}\}$ such that the following properties hold*:

(i) *The height h_ξ of each T-tower $\xi \in \Xi$ is greater than $n + m$.*

(ii) $$\mu_j\left(\bigcup_{\xi\in\Xi}\left(\bigcup_{i=0}^{n-1} T^i B_\xi \cup \bigcup_{i=1}^{m} T^{h_\xi - i} B_\xi\right)\right) < \varepsilon, \quad j = 1, \dots, k,$$
where B_ξ is the base of $\xi \in \Xi$.

Proof. The proof can be deduced from [3, Theorem 2.5]. □

2.13. Theorem. *Let $S \in \mathcal{A}p$. Then for any $R \in \mathcal{A}p$ and any τ-neighborhood $U(R) = U(R; \mu_1, \dots, \mu_p; \varepsilon)$, there exists $T \in Aut(X, \mathcal{B})$ such that $T^{-1}ST \in U(R)$. In other words, $\overline{\{T^{-1}ST\colon T \in Aut(X, \mathcal{B})\}}^{\tau} = \mathcal{A}p$.*

Proof. Apply **2.12** to $R \in \mathcal{A}p$, $\mu_1, \dots, \mu_p \in \mathcal{M}_1(X)$, $\varepsilon/2 > 0$, and $n = m = 2$. We obtain a partition of X into R-towers $\Xi = \{\xi_k\colon k \in \mathbb{N}\}$ satisfying (i), (ii). Choose a sufficiently large $K \in \mathbb{N}$ such that $\mu_j(\bigcup_{k>K} C_k) < \varepsilon/2$, $j = 1, \ldots, p$, where C_k is the set supporting the tower ξ_k. Therefore, we have that

$$\mu_j\left(\bigcup_{k=1}^{K} \bigcup_{i=2}^{h_k-3} T^i B_k\right) > 1 - \varepsilon$$

for all $j = 1, \ldots, p$, where B_k is the base of ξ_k and h_k is its height.

Since S is aperiodic, we can find K disjoint S-towers $\Lambda = \{\lambda_l\colon k = 1, \ldots, K\}$ such that the height of λ_k is h_k and λ_k has the same cardinality as ξ_k for all $k = 1, \ldots, K$. Denote by Z_k the base of λ_k and let $D_k = \bigcup_{i=0}^{h_k-1} T^i Z_k$ be the support of λ_k. Let Q_k be a Borel isomorphism which maps B_k onto Z_k, $k = 1, \ldots, K$, and let Q be a Borel isomorphism which sends $X - (C_1 \cup \cdots \cup C_K)$ onto $X - (D_1 \cup \cdots \cup D_K)$. Define the automorphism T as follows:

$$Tx = \begin{cases} S^i Q_k R^{-i}, & \text{if } x \in R^i B_k,\ 0 \leqslant i \leqslant h_k - 1,\ 1 \leqslant k \leqslant K, \\ Qx, & \text{if } x \notin C_1 \cup \cdots \cup C_K. \end{cases}$$

Then T is defined everywhere on the set X. It is not hard to see that

$$\{x \in X\colon Rx = T^{-1}STx \text{ and } R^{-1}x = T^{-1}S^{-1}Tx\} \supset \bigcup_{k=1}^{K} \bigcup_{i=2}^{h_k-3} R^i B_k.$$

Hence $\mu_j(E(R, T^{-1}ST)) < \varepsilon$, $j = 1, \ldots, p$, and therefore $T^{-1}ST \in U(R)$. □

3. PATH-CONNECTEDNESS OF $(Aut_0(X, \mathcal{B}), \tau_0)$

3.1. In this section, we prove that $Aut_0(X, \mathcal{B})$ is path-connected in the topology τ_0. We first show that the group $Aut(X, \mathcal{B})$ does not possess this property.

3.2. Proposition. *The topological group $(Aut(X, \mathcal{B}), \tau)$ is not path-connected.*

Proof. Let P be an arbitrary involution in $Aut(X, \mathcal{B})$, that is $Px \neq x$ and $P^2x = x$ for all $x \in X$. We will show that P cannot be connected with the identity $\mathbb{I}$ by a continuous path, i.e., there exists no continuous map $f\colon [0, 1] \to Aut(X, \mathcal{B})$ such that $f(0) = \mathbb{I}$, $f(1) = P$. Assume that the converse is true and let f be such a path. Choose x_0, y_0 in X such that $Px_0 = y_0$, $Py_0 = x_0$. Consider the τ-neighborhood $U(P) = U(P; \delta_{x_0}, \delta_{y_0}; 1/2)$ of P. Notice that $U(P)$ contains only those automorphisms from $Aut(X, \mathcal{B})$ which map x_0 to y_0 and y_0 to x_0. Since, by assumption, f is continuous, there exists $t^* \in (0, 1)$ such that $f((t^*, 1]) \subset U$. Set

$$t_0^* = \inf\{t \in [0, 1]\colon f(s) \in U,\ t \leqslant s \leqslant 1\}.$$

Clearly, $0 \leqslant t_0^* < 1$. Consider now the neighborhood $U(f(t_0^*)) = U(f(t_0^*); \delta_{x_0}, \delta_{y_0}; 1/2)$ of $f(t_0^*)$. If $t_0^* > 0$, then there exist α and β such that $\alpha < t_0^* < \beta$ and $f([\alpha, \beta]) \subset U(f(t_0^*))$. We obtain that $f(\beta) \in U(f(t_0^*)) \cap U(P)$ and therefore, $f(t_0^*)x_0 = f(\beta)x_0 = Px_0 = y_0$ and $f(t_0^*)y_0 = f(\beta)y_0 = Py_0 = x_0$. A similar relation holds for $f(\alpha)$. Thus, $f(t_0^*) \in U(P)$ and therefore t_0^* cannot be the infimum. Hence $t_0^* = 0$ and $\mathbb{I} = f(0) \in U(P)$, which is a contradiction. $\square$

3.3. Remark. Let $T \in Aut(X, \mathcal{B})$ and let $A \in \mathcal{B}$ be a complete T-section such that every point from A is recurrent. If a Borel set B contains A, then B is also a complete T-section which consists of recurrent points.

3.4. Theorem. *The topological group* $(Aut_0(X, \mathcal{B}), \tau_0)$ *is path-connected.*

Proof. We first prove separately that every periodic automorphism P and every aperiodic automorphism T can be connected with the identity by a continuous path (see **3.5** and **3.6** respectively). By **1.4**, these two results will give the proof for any automorphism. Recall that, by **1.13**, it is sufficient to deal with continuous measures only. $\square$

3.5. Lemma. *Let* $P \in Aut_0(X, \mathcal{B})$ *be a periodic automorphism. Then there exists a continuous map* $f : [0, 1] \to (Aut_0(X, \mathcal{B}), \tau_0)$ *such that* $f(0) = \mathbb{I}$ *and* $f(1) = P$.

Proof. By **1.4**, we have the decomposition of $X = \bigcup_{k \geqslant 1} X_k$ where $X_k = \bigcup_{i=0}^{k-1} P^i B_k$ is a P-tower. Without loss of generality, we can assume that the B_k's are uncountable Borel sets and therefore they all are isomorphic to the unit interval $(0, 1)$. Let $\psi_k : (0, 1) \to B_k$, $k \in \mathbb{N}$, be a Borel isomorphism. For each B_k, define the map $\Psi_k : [0, 1] \to \mathcal{B} \restriction B_k$ by

$$\Psi_k(t) = \begin{cases} \emptyset, & t = 0, \\ \psi_k((0, t)), & t \in (0, 1), \\ B_k, & t = 1. \end{cases}$$

Claim 1. The function $t \mapsto \Psi_k(t)$, $k \in \mathbb{N}$, is continuous on $[0, 1]$ in the sense that for any non-atomic $\mu \in \mathcal{M}_1(X)$,

$$\lim_{t \to t_0} \mu\big(\Psi_k(t) \Delta \Psi_k(t_0)\big) = 0, \quad t_0 \in [0, 1].$$

The proof is straightforward.

Define now the path $f : [0, 1] \to Aut_0(X, \mathcal{B})$ as follows:

$$f(t)x = \begin{cases} Px, & \text{if } x \in \bigcup_{i=0}^{k-1} P^i \Psi_k(t) \text{ for some } k, \\ x, & \text{otherwise.} \end{cases}$$

It is clear that $f(0) = \mathbb{I}$ and $f(1) = P$ and we need to show only that $f(t)$ is continuous. To do this, fix $t_0 \in [0, 1]$ and consider the map $\Theta : t \mapsto \mu(E_0(f(t), f(t_0)))$ on $[0, 1]$ where $\mu \in \mathcal{M}_1(X)$ is non-atomic and E_0 is defined in **1.10**.

Given $\varepsilon > 0$, choose $K > 0$ such that $\mu(\bigcup_{k > K} X_k) < \varepsilon/2$. Therefore $\mu(E_0(f(t), f(t_0))) \leqslant \mu(E_0(f(t), f(t_0)) \cap \bigcup_{k=1}^{K} X_k) + \varepsilon/2$. We see that

$$\mu\left(E_0\big(f(t), f(t_0)\big) \cap \bigcup_{k=1}^{K} X_k\right) = \sum_{k=1}^{K} \sum_{i=0}^{k-1} \mu\big(P^i\big(\Psi_k(t) \Delta \Psi_k(t_0)\big)\big).$$

The fact that Θ is continuous follows from Claim 1.

If now $U_0(f(t_0); \mu_1, \ldots, \mu_n; \varepsilon)$ is a τ_0-neighborhood of $f(t_0)$, then we apply the proved result to each measure μ_i. The lemma is proved. $\square$

3.6. Lemma. *Let $T \in Aut_0(X, \mathcal{B})$ be an arbitrary aperiodic automorphism. Then there exists a continuous map $P : [0,1] \to (Aut_0(X, \mathcal{B}), \tau_0)$ such that $P(0) = \mathbb{I}$ and $P(1) = T$. Moreover, for all $t \neq 1$, the automorphism $P(t)$ is periodic.*

Proof. By **1.7**, choose a vanishing sequence of markers $\{A_n\}_{n=0}^{\infty}$ with $A_0 = X$. Without loss of generality, we can assume that the set $F_n := A_n \setminus A_{n+1}$ is uncountable for all n. Take a sequence of real numbers $\{t_n\}$ such that $0 = t_0 < t_1 < t_2 < \cdots < 1$ and $\lim_{n\to\infty} t_n = 1$. Let $\psi_n : [t_n, t_{n+1}) \to F_n$, $n \in \mathbb{N}$, be a Borel isomorphism. Define the function $\Phi : [0,1] \to \mathcal{B}$ as follows:

$$\Phi(t) = \begin{cases} A_n, & \text{if } t = t_n,\ n \geqslant 0, \\ A_n - \psi_n([t_n, t)), & \text{if } t \in (t_n, t_{n+1}),\ n \geqslant 0, \\ \emptyset, & \text{if } t = 1. \end{cases}$$

Observe that for each $t \in [0,1)$ there exists $n = n(t) \in \mathbb{N}$ such that $\Phi(t) \supset A_n$. By **3.3**, we get that $\Phi(t)$ is a T-section which consists of recurrent points. We also notice that $\lim_{t\to s} \mu(\Phi(t) \Delta \Phi(s)) = 0$ for any non-atomic $\mu \in \mathcal{M}_1(X)$ and $s \in [0,1]$.

Now, we apply the method of the proof of the Rokhlin lemma [3] to produce a continuous family $\{P(t)\}$ of periodic automorphisms which approximates T.

Fix $t \in [0,1]$. By **1.5**, define the set $\Phi_n(t) = \{x \in \Phi(t)\colon\ T^n x \in \Phi(t),\ T^i x \notin \Phi(t) \text{ for } 1 \leqslant i \leqslant n-1\}$, $n \in \mathbb{N}$, where $\Phi_n(1) = \emptyset$ and $\Phi_n(0) = X$. Clearly, $\Phi_n(t)$ may be empty for some n. By construction, the entire space X is partitioned into T-towers with bases $\Phi_n(t)$.

Define

$$P(t) = \begin{cases} T^{-n+1}x, & \text{whenever } x \in T^{n-1}\Phi_n(t) \text{ for some } n \in \mathbb{N}, \\ Tx, & \text{otherwise.} \end{cases}$$

Notice that $P(0) = \mathbb{I}$, $P(1) = T$, and $P(t)$ is periodic if $t \neq 1$.

It remains to prove that the map $P : t \mapsto P(t)$ sending $[0,1]$ to $Aut_0(X, \mathcal{B})$ is continuous. Note that for every $n \in \mathbb{N}$ the family $\{\Phi_n(t)\}$ is continuous, i.e., for a non-atomic measure $\mu \in \mathcal{M}_1(X)$,

$$\lim_{t\to s} \mu\big(\Phi_n(t) \Delta \Phi_n(s)\big) = 0.$$

Indeed, this fact follows from continuity of $\Phi(t)$ and from the relation

$$\Phi_n(t) = \big(\Phi(t) \cap T^{-n}\Phi(t)\big) \setminus \bigcup_{i=1}^{n-1} T^{-i}\Phi(t).$$

Next, show that for $t', t'' \in [0,1]$ we have

$$E_0(P(t'), P(t'')) = \{x \in X\colon\ P(t')x \neq P(t'')x\}$$
$$\subset \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{n-1} T^k(\Phi_n(t') \Delta \Phi_n(t'')).$$

Let $t' < t''$ for definiteness. Suppose that $x \in E_0(P(t'), P(t''))$. Then x belongs to a T-tower constructed over $\Phi(t')$, that is $x \in T^l \Phi_n(t')$ for some $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n-1$. If $x \in T^k \Phi_n(t'')$, then by construction of $P(t')$ and $P(t'')$, we have that $P(t')x = P(t'')x$. Therefore, $x \in T^k(\Phi_n(t') \setminus \Phi_n(t''))$.

Fix $s \in [0,1]$. Consider a neighborhood $U(P(s)) = U(P(s); \mu_1, \dots, \mu_m; \varepsilon)$ of $P(s)$ where measures $\mu_1, \dots, \mu_m$ are continuous. By definition of $P(s)$, we have that

$$X = \bigcup_{n \geqslant 1} \bigcup_{k=0}^{n-1} P^k(s)\Phi_n(s) = \bigcup_{n \geqslant 1} \bigcup_{k=0}^{n-1} T^k \Phi_n(s),$$

and these unions are disjoint. Take $N \geqslant 1$ such that

$$(*) \qquad \mu_i\left(\bigcup_{n \geqslant N} \bigcup_{k=0}^{n-1} T^k \Phi_n(s) \right) < \varepsilon/8$$

for $1 \leqslant i \leqslant m$. By continuity of $\{\Phi_n(t)\}$, we can find a neighborhood $O(s) \subset [0,1]$ such that for any $t \in O(s)$ one has

$$(**) \qquad \mu_i\big(T^k(\Phi_n(t) \Delta \Phi_n(s))\big) < \frac{\varepsilon}{8N^2}$$

for $1 \leqslant i \leqslant m$, $1 \leqslant n \leqslant N-1$, and $0 \leqslant k \leqslant n-1$.

It follows from $(*)$ and $(**)$ that

$$\mu_i\left(\bigcup_{n=1}^{N-1} \bigcup_{k=0}^{n-1} T^k \Phi_n(t) \right) > 1 - \varepsilon/4,$$

hence $\mu_i(\bigcup_{n \geqslant N} \bigcup_{k=0}^{n-1} T^k \Phi_n(t)) < \varepsilon/4$ when $i = 1, \dots, m$ and $t \in O(s)$.

Therefore, we have

$$\mu_i\left(\bigcup_{n \geqslant N} \bigcup_{k=0}^{n-1} T^k\big(\Phi_n(t) \Delta \Phi_n(s)\big) \right)$$
$$\leqslant \mu_i\left(\bigcup_{n \geqslant N} \bigcup_{k=0}^{n-1} T^k \Phi_n(s) \right) + \mu_i\left(\bigcup_{n \geqslant N} \bigcup_{k=0}^{n-1} T^k \Phi_n(t) \right)$$
$$< \varepsilon/2,$$

for all $t \in O(s)$. Thus, for all $t \in O(s)$, we obtain that

$$
\begin{aligned}
\mu_i\big(E_0(P(t),P(s))\big) &\leqslant \mu_i\left(\bigcup_{n=1}^{\infty}\bigcup_{k=0}^{n-1} T^k\big(\Phi_n(t)\Delta\Phi_n(s)\big)\right)\\
&\leqslant \mu_i\left(\bigcup_{n=1}^{N-1}\bigcup_{k=0}^{n-1} T^k\big(\Phi_n(t)\Delta\Phi_n(s)\big)\right)\\
&\quad+\mu_i\left(\bigcup_{n=N}^{\infty}\bigcup_{k=0}^{n-1} T^k\big(\Phi_n(t)\Delta\Phi_n(s)\big)\right)\\
&\leqslant \sum_{n=1}^{N-1}\sum_{k=0}^{n-1}\frac{\varepsilon}{8N^2}+\frac{\varepsilon}{2}\\
&< \varepsilon.
\end{aligned}
$$

This means that $\{P(t)\colon t \in O(s)\} \subset U(P(s))$ and the proof is completed. □

3.7. Let $T \in Aut(X, \mathcal{B})$ and let $Orb_T(x)$ denote the T-orbit of $x \in X$. Recall the definition of the *full group* $[T]$ generated by $T \in Aut(X, \mathcal{B})$:

$$[T] = \big\{\gamma \in Aut(X, \mathcal{B}) \mid \gamma x \in Orb_T(x),\ \forall x \in X\big\}.$$

Then every $\gamma \in [T]$ defines a Borel function $m_\gamma : X \to \mathbb{Z}$ such that $\gamma x = T^{m_\gamma(x)}x$, $x \in X$. It follows easily from **1.9** that $[T]$ is τ-closed in $Aut(X, \mathcal{B})$.

Note that if $T \sim S$, then $T^n \sim S^n$, $\forall n \in \mathbb{Z}$. Therefore $Orb_T(x) = Orb_S(x)$ everywhere except a countable set. This means that one can extend the definition of full group to automorphisms from $Aut_0(X, \mathcal{B})$.

3.8. Corollary. *The full group $[T]$ of any $T \in Aut_0(X, \mathcal{B})$ is path-connected.*

Proof. The proofs of **3.5** and **3.6** show that the constructed paths $f(t)$ and $P(t)$ connecting the identity with T belong to the full group $[T]$. □

ACKNOWLEDGEMENTS

We are thankful to A.H. Dooley, A. Kechris, J. Kwiatkowski, and B. Miller for numerous discussions of these results. The first-named author thanks the University of New South Wales for the warm hospitality and the Australian Research Council for its support.

REFERENCES

[1] Alpern S., Prasad V.S. – Typical Dynamics of Volume Preserving Homeomorphisms, Cambridge Univ. Press, Cambridge, UK, 2000.

[2] Becker H., Kechris A. – The Descriptive Set Theory of Polish Group Actions, London Math. Soc. Lecture Notes Series, vol. 232, Cambridge Univ. Press, Cambridge, UK, 1996, xii + 136 pp.

[3] Bezuglyi S., Dooley A.H., Kwiatkowski J. – Topologies on the group of Borel automorphisms of a standard Borel space, Preprint, 2003.

[4] Bezuglyi S., Dooley A.H., Kwiatkowski J. – Topologies on the group of homeomorphisms of a Cantor set, Preprint, 2004.

[5] Bezuglyi S., Dooley A.H., Medynets K. – The Rokhlin lemma for homeomorphisms of a Cantor set, Proc. AMS (to appear).

[6] Bezuglyi S., Golodets V. – The topological properties of full groups of transformations of a measure space, Sib. Math. J. **21** (1980) 3–21.

[7] Bezuglyi S., Kwiatkowski J. – The topological full groups of a Cantor minimal system is dense in the full group, Topol. Methods in Nonlinear Anal. **16** (2000) 371–397.

[8] Bezuglyi S., Kwiatkowski J. – Topologies on full groups and normalizers of Cantor minimal systems, Math. Phys. Anal. Geom. **9** (3) (2002) 1–10.

[9] Cornfeld I., Fomin S., Sinai Ya. – Ergodic Theory, Springer-Verlag, Berlin/New York, 1982.

[10] Danilenko A. – The topological structure of Polish groups and groupoids of measure space transformations, Publ. RIMS, Kyoto University **31** (1995) 913–940.

[11] Glasner E., King J. – A zero-one law for dynamical properties, in: Topological Dynamics and Applications, in: Contemp. Math., vol. 215, 1998, pp. 215–242.

[12] Glasner E., Weiss B. – The topological Rokhlin property and topological entropy, Amer. J. Math. **123** (2001) 1055–1070.

[13] Halmos P. – Lectures on Ergodic Theory, Publications of the Mathematical Society of Japan, vol. 3, The Mathematical Society of Japan, 1956, vii + 99 pp.

[14] Keane M. – Contractibility of the automorphism group of a nonatomic measure space, Proc. Amer. Math. Soc. **26** (1970) 420–422.

[15] Kechris A., Miller B. – Topics in orbit equivalence theory, Preprint, 2003.

[16] Miller B. – Private communication.

[17] Nadkarni M.G. – Basic Ergodic Theory, second ed., Birkhäuser, Basel, 1998.

[18] Rokhlin V.A. – Selected topics from the metric theory of dynamical systems, Uspekhi Matem. Nauk (N.S.) **4** (2) (1949) 57–128 (in Russian).

[19] Shortt R.M. – Normal subgroups of measurable automorphisms, Fund. Math. **135** (1990) 177–187.

(Received January 2004)